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Third party stabilization of unstable coordination in systems of coupled oscillators

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Abstract. The Haken-Kelso-Bunz (HKB) system of equations is a well-developed model for dyadic rhythmic coordination in biological systems. It captures ubiquitous empirical observations of bistability – the coexistence of in-phase and antiphase motion – in neural, behavioral, and social coordination. Recent work by Zhang and colleagues has generalized HKB to many oscillators to account for new empirical phenomena observed in multiagent interaction. Utilising this generalization, the present work examines how the coordination dynamics of a pair of oscillators can be augmented by virtue of their coupling to a third oscillator. We show that stable antiphase coordination emerges in pairs of oscillators even when their coupling parameters would have prohibited such coordination in their dyadic relation. We envision two lines of application for this theoretical work. In the social sciences, our model points toward the development of intervention strategies to support coordination behavior in heterogeneous groups (for instance in gerontology, when younger and older individuals interact). In neuroscience, our model will advance our understanding of how the direct functional connection of mesoscale or microscale neural ensembles might be switched by their changing coupling to other neural ensembles. Our findings illuminate a crucial property of complex systems: how the whole is different than the system's parts.

Keywords: coordination dynamics, HKB, multistability, attractor, synergetics, complex systems.

1. Introduction

The Haken-Kelso-Bunz (HKB) equation [1] is one of the most well-understood mathematical models in the field of synergetics. It was developed to model an observed nonlinear phase transition in human motor behavior [2]. A human subject's simultaneously wiggled left and right index fingers were observed to coordinate in-phase or antiphase at low frequencies (in-phase: both fingers rise and fall together; antiphase: one finger rises as the other falls). But at high frequencies, only in-phase was stable. Experiments and model established a bifurcation from bistability (in-phase and antiphase) to monostability (only in-phase). Annihilation of the



antiphase attractor was found to depend on both movement frequency and coupling strength parameters. Beyond motor behavior, the model has found a comprehensive empirical ground in biological coordination within and across individuals and species, at neural, behavioral and social levels (see [3] and [4] for reviews).

Though the HKB model was originally formulated to describe a pair of oscillators, it has recently been generalized to systems of $N + 1$ oscillators by Zhang et al [5] (see also [6] and [7] for related work). This multiadic perspective has opened new avenues to study how the dyadic relationship between a set of coordinating elements (e.g. two people) is affected by a third party (e.g. an extra social partner).

In this paper, we study a system of $N + 1 = 3$ oscillators, whose pairwise interactions are governed by coupling parameters that either allow or prohibit bistability. We visualize the system's relative phase potential function as a 2D landscape where local minima denote stable fixed points or attractors (the possible coordination states of the system). We find that coupling a third oscillator to a pair of coupled oscillators can cause the antiphase fixed point between the original pair to become stable even when it would have been unstable for that original pair in isolation, and hence bistability has been restored through coupling of the third oscillator.

2. Generalized HKB Systems

The HKB model [1] arises as an approximation to the dynamics of a system of two coupled, hybrid Rayleigh-Van der Pol oscillators. The underlying Rayleigh-Van der Pol model is readily extended to arbitrarily many oscillators [8] with the evolution equations

$$\ddot{x}_i - (\alpha_i - \beta_i \dot{x}_i^2 - \gamma_i x_i^2) \dot{x}_i + \omega_i^2 x_i = - \sum_{j=0}^N (a_{ij} - b_{ij} (\dot{x}_i - \dot{x}_j)^2 - c_{ij} (x_i - x_j)^2) (\dot{x}_i - \dot{x}_j). \quad (2.1)$$

Here, $x_i(t)$ denotes the position of the i^{th} oscillator at time t , with $i = 0, \dots, N$, the parameters $(\alpha_i, \beta_i, \gamma_i, \omega_i)$ describe its autonomous dynamics, and the parameters (a_{ij}, b_{ij}, c_{ij}) describe its coupling to the j^{th} oscillator. We assume that all of these parameters are positive. Note that the effective damping term on the left is then negative in the limit of slow motions ($\beta_i \dot{x}_i^2 \ll \alpha_i$) with small amplitude ($\gamma_i x_i^2 \ll \alpha_i$). Each oscillator in isolation therefore exhibits a self-excitatory dynamics in that regime. Its dynamics becomes self-regulatory, however, as the amplitude and/or speed increases, and the net effect is that each of these oscillators in isolation tends toward a nearly harmonic motion characterized by a specific frequency $\Omega \sim \omega_i$ and amplitude $r_i \sim \frac{4\alpha_i}{3\beta_i\Omega^2 + \gamma_i}$. However, the coupling terms permit other stable, collective motions of the system in which the individual oscillators' motions deviate from their preferred, autonomous states. The HKB model aims to identify and characterize such collective states of motion.

The HKB approximation to eq. (2.1) emerges in several stages. The process begins by recasting eq. (2.1) in terms of an amplitude and phase of each oscillator's motion, setting

$$x_i(t) =: r_i(t) \cos(\Omega t + \theta_i(t)) \quad \text{and} \quad \dot{x}_i(t) =: -\Omega r_i(t) \sin(\Omega t + \theta_i(t)), \quad (2.2)$$

where Ω is an entrainment frequency characteristic of some bulk, oscillatory motion of the entire system yet to be determined [9]. One then specializes to motions in which the anomalous phase $\theta_i(t)$ of each oscillator varies only slightly over the period $T := 2\pi/\Omega$ of a single bulk oscillation, and its amplitude $r_i(t)$ varies more slowly still. Finally, one replaces eq. (2.1) with its average over a single such period T (a Krylov–Bogoliubov approximation) [10], and treats the amplitudes r_i as time-independent constants to find the approximate evolution equations

$$\dot{\theta}_i = \frac{\omega_i^2 - \Omega^2}{2\Omega} - \sum_{j \neq i} A_{ij} \sin(\theta_i - \theta_j) - \sum_{j \neq i} 2B_{ij} \sin 2(\theta_i - \theta_j), \quad (2.3)$$

where the effective coupling parameters are

$$A_{ij} := \frac{a_{ij}}{2} \frac{r_j}{r_i} - 2 \frac{r_i^2 + r_j^2}{r_i r_j} B_{ij} \quad \text{with} \quad B_{ij} := \frac{2b_{ij}\Omega^2 + c_{ij}}{16} r_j^2. \quad (2.4)$$

This multi-oscillator phase evolution equation was proposed by Zhang et al [5] as a generalization of the HKB dynamics to larger systems of oscillators.

In the case of two identical oscillators, the evolution equations from eq. (2.3) combine to yield a single, self-contained equation for their relative phase:

$$\dot{\theta}_1 - \dot{\theta}_0 = -A \sin(\theta_1 - \theta_0) - 2B \sin 2(\theta_1 - \theta_0), \quad (2.5)$$

where $A := A_{10} + A_{01}$ and $B := B_{10} + B_{01}$. This is the original HKB equation. Its key feature is that it can be either *bistable*, with stable fixed points corresponding to both in-phase ($\theta_1 = \theta_0$) and anti-phase ($\theta_1 - \theta_0 = \pi$) motions of the oscillators if $A < 4B$, or *monostable*, with only the in-phase motion being stable if $A > 4B$. This dichotomy mirrors the qualitative features of empirical observations of bimanual coordination in humans [2].

Returning to the case of a general system of $N + 1$ oscillators, the analysis of the fixed points of eq. (2.3) is greatly simplified if there exists a potential function $V(\theta_0, \dots, \theta_N)$ such that

$$\dot{\theta}_i = -\frac{\partial V}{\partial \theta_i}(\theta_0, \dots, \theta_N). \quad (2.6)$$

Such a potential does exist when the coupling matrices are both *symmetric*¹:

$$A_{ij} = A_{ji} \quad \text{and} \quad B_{ij} = B_{ji}. \quad (2.7)$$

The potential in this symmetric-coupling case is

$$V(\theta_0, \dots, \theta_N) = -\sum_i \frac{\omega_i^2 - \Omega^2}{2\Omega} \theta_i - \sum_{i < j} A_{ij} \cos(\theta_j - \theta_i) - \sum_{i < j} B_{ij} \cos 2(\theta_j - \theta_i), \quad (2.8)$$

where the last two terms feature double sums over both j and $i < j$. The stable fixed points of eq. (2.3) coincide with the local minima of this potential.

Coordination states of the generalized HKB system correspond to fixed points, not of the *absolute* phases whose evolution is described by eq. (2.3), but rather of the oscillators' *relative* phases. Accordingly, it is useful to switch the coordinates we use on phase space to be relative phases themselves. We will choose the relative phase coordinates

$$\Phi := \sum_{i=0}^N \theta_i \quad \text{and} \quad \phi_\alpha := \theta_\alpha - \theta_0, \quad (2.9)$$

with $\alpha = 1, \dots, N$ in the latter case. Note that we switch to Greek indices, rather than Latin, to help emphasize the difference in counting of the $N + 1$ absolute phase coordinates ($\theta_0, \dots, \theta_N$)

¹ One can still define a potential for almost all cases of non-symmetric couplings. Doing so adds some technical complication, however, and sheds little additional light on our qualitative results below. We therefore restrict to the simpler case of symmetric couplings in what follows for convenience. The key feature the coupling matrices must have in order for a potential to exist is that the Jacobian matrix $\partial \dot{\theta}_i / \partial \theta_j$ for the dynamics of eq. (2.3) should have a complete basis of eigenvectors throughout phase space.

and of the N relative phase coordinates (ϕ_1, \dots, ϕ_N) throughout the computations that follow. One can show that the potential of eq. (2.8) takes the form

$$V(\Phi, \phi_1, \dots, \phi_N) = \frac{\Omega^2 - \omega_{\text{rms}}^2}{2\Omega} \Phi - \sum_{\alpha} \left(\frac{\omega_{\alpha}^2 - \omega_{\text{rms}}^2}{2\Omega} \phi_{\alpha} + A_{0\alpha} \cos \phi_{\alpha} + B_{0\alpha} \cos 2\phi_{\alpha} \right) - \sum_{\alpha < \beta} \left(A_{\alpha\beta} \cos(\phi_{\beta} - \phi_{\alpha}) + B_{\alpha\beta} \cos 2(\phi_{\beta} - \phi_{\alpha}) \right) \quad (2.10)$$

in these relative-phase coordinates, where ω_{rms} denotes the root-mean-square average of the individual oscillators' characteristic frequencies ω_i (with $i = 0, \dots, N$).

One drawback of casting the potential in relative-phase coordinates, however, is the need to account for the coordinate transformation in formulating the equations of motion. Specifically, the system's dynamics generally does not take the simple gradient form of eq. (2.6) in arbitrary coordinates. Rather, for a *general* transformation $(\theta_0, \dots, \theta_N) \mapsto (\Phi, \phi_1, \dots, \phi_N)$, applying the chain rule of multi-variate calculus to eq. (2.6) gives

$$\dot{\phi}_{\alpha} = - \sum_i \frac{\partial \phi_{\alpha}}{\partial \theta_i} \frac{\partial V}{\partial \theta_i} = - \sum_i \frac{\partial \phi_{\alpha}}{\partial \theta_i} \left(\frac{\partial \Phi}{\partial \theta_i} \frac{\partial V}{\partial \Phi} + \sum_{\beta} \frac{\partial \phi_{\beta}}{\partial \theta_i} \frac{\partial V}{\partial \phi_{\beta}} \right), \quad (2.11)$$

with a similar expression of $\dot{\Phi}$. For the *specific* coordinate transformation of eq. (2.9), this general expression reduces to

$$\dot{\phi}_{\alpha} = - \frac{\partial V}{\partial \phi_{\alpha}} - \sum_{\beta} \frac{\partial V}{\partial \phi_{\beta}} \quad \text{and} \quad \dot{\Phi} = -(N+1) \frac{\partial V}{\partial \Phi}. \quad (2.12)$$

One must be careful to include the second term in the evolution equation for the relative phases when computing fixed points and, especially, when assessing their stability or lack thereof. It is convenient to write the relative phase dynamics from this equation in the metric form

$$\dot{\phi}_{\alpha} = - \sum_{\beta} g_{\alpha\beta} \frac{\partial V}{\partial \phi_{\beta}} \quad \text{with} \quad g_{\alpha\beta} := \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}. \quad (2.13)$$

Note that the evolution of the ‘‘center-of-mass’’ phase Φ is trivial, $\dot{\Phi} = (N+1) \frac{\omega_{\text{rms}}^2 - \Omega^2}{2\Omega}$, as indeed can be confirmed directly from eqs. (2.3) and (2.9). The evolution of the relative phase coordinates ϕ_{α} is also independent of Φ , generalizing the self-contained HKB equation for the relative phase in a two-oscillator system. Accordingly, we henceforth drop the Φ coordinate entirely.

3. Induced Dyadic Bistability in Many-Oscillator Systems

We now explore how the stability properties of a given pair of HKB oscillators can be influenced by their mutual coupling to other such oscillators. For concreteness, we will consider the case of three oscillators (i.e., $N = 2$ in the notation of the previous section) having a common natural frequency $\omega_i = \Omega$ and the symmetric couplings $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ discussed above. The relative-phase potential of eq. (2.10) is then

$$V(\phi_1, \phi_2) = -A_{01} \cos \phi_1 - A_{02} \cos \phi_2 - A_{12} \cos(\phi_1 - \phi_2) - B_{01} \cos 2\phi_1 - B_{02} \cos 2\phi_2 - B_{12} \cos 2(\phi_1 - \phi_2). \quad (3.1)$$

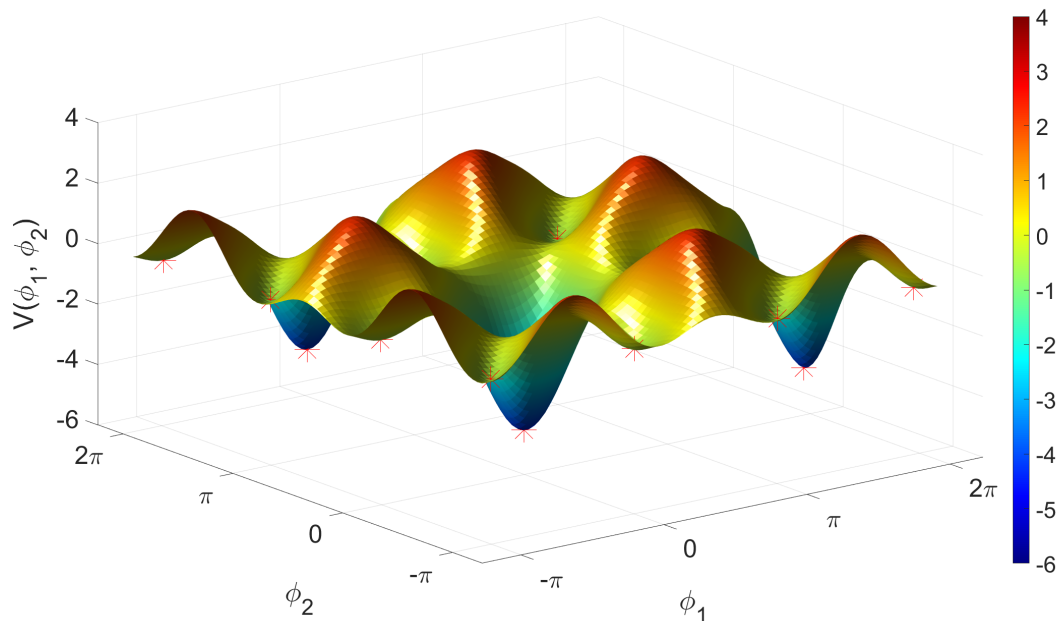


Figure 1. A plot of the relative phase potential function landscape for $A_{ij} = 2B_{ij} = 1$ for each i, j . Note the many valleys (marked with red asterisks) in which an oscillator moving around in this landscape will become trapped. These valleys are the local minima corresponding to the coordination states. There are two types of valleys in this landscape: in-phase valleys, which have relatively very deep and wide basins of attraction, and antiphase valleys, which are narrower and shallower, reflecting the fact that the in-phase state is more stable than the antiphase state. Each of these valleys is separated by a distance of π , and repeats infinitely on the potential surface in a 2π -periodic pattern.

Figure 1 presents a landscape plot for a typical set of coupling parameter values ($A_{ij} = 1$ and $B_{ij} = \frac{1}{2}$ for all $0 \leq i, j \leq 2$), marking the local minima that correspond to stable, fixed-point motions of the system. The independent variables ϕ_α are periodic in this plot, so a single, physical fixed point may be represented by multiple local minima. There are four physically distinct fixed-point motions. The first corresponds to $(\phi_1, \phi_2) = (0, 0)$, meaning that all three oscillators move in-phase as a group. The others correspond to $(\phi_1, \phi_2) \in \{(0, \pi), (\pi, 0), (\pi, \pi)\}$, meaning that one of the three possible pairs of oscillators move in-phase with one another, but anti-phase to the third oscillator. This array of fixed points is not at all surprising in this instance, however, since the coupling parameters satisfy $A_{ij} = 2B_{ij} < 4B_{ij}$ for each pair of oscillators. This means that each pair of oscillators in this system, if it were to be isolated from the third, would exhibit a *bistable* HKB dynamics, and the four fixed points of this triadic system correspond to the different choices of the two possible fixed points for each of the pairs.

We now contrast the fixed-point analyses for four qualitatively different triads of oscillators. In the first, all three dyads within the triad are coupled bistably, having parameter values

$$(A_{01}, B_{01}) = (A_{02}, B_{02}) = (A_{12}, B_{12}) = (A_{bi}, B_{bi}), \quad (3.2a)$$

with $(A_{bi}, B_{bi}) := (1, 1)$. In the second triad, one dyadic coupling is changed to be monostable, while the other two dyads retain their bistable couplings:

$$(A_{01}, B_{01}) = (A_{mon}, B_{mon}) \quad \text{and} \quad (A_{02}, B_{02}) = (A_{12}, B_{12}) = (A_{bi}, B_{bi}), \quad (3.2b)$$

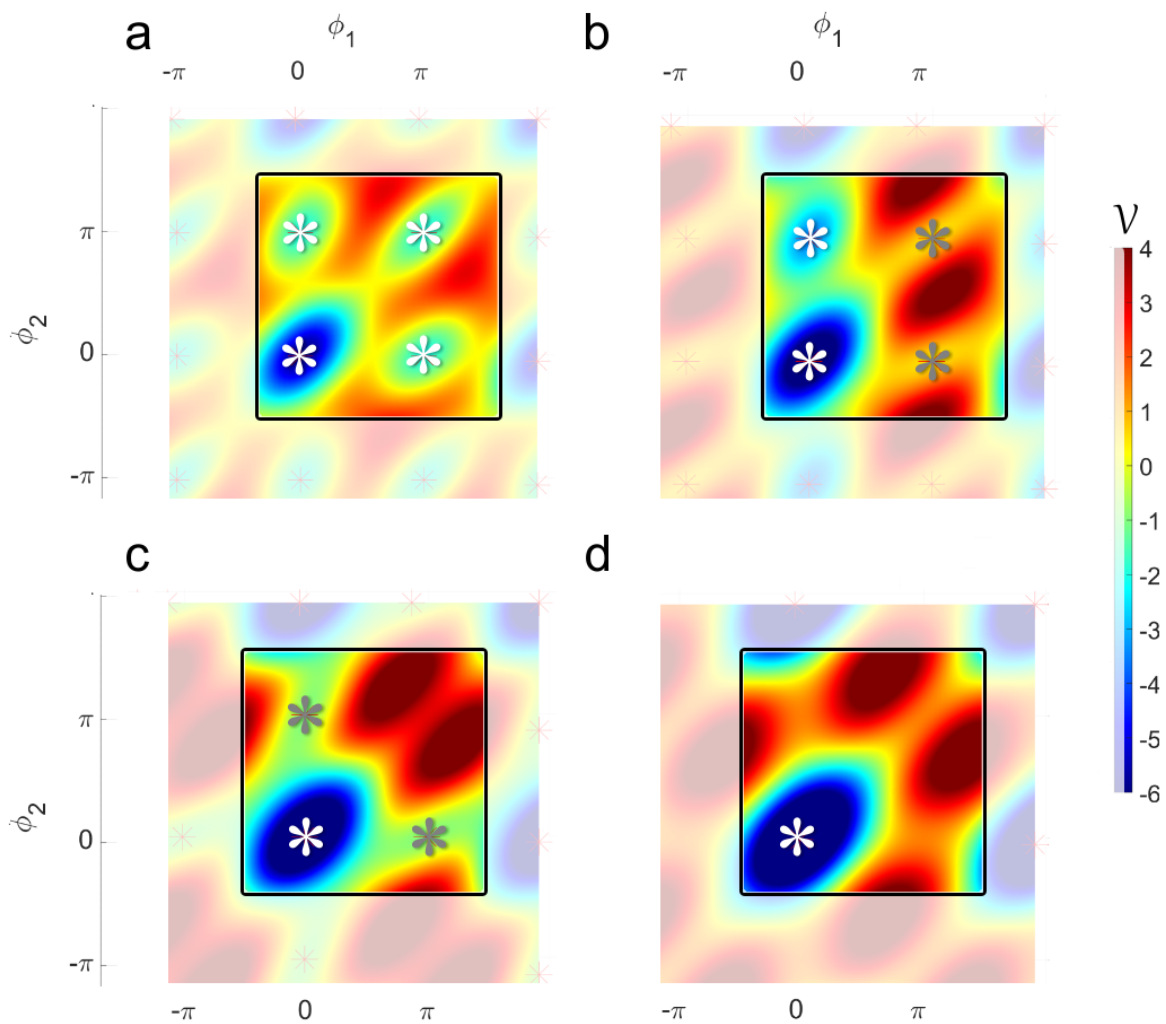


Figure 2. Top-down views of the potential function used to visualize stable fixed points (marked with asterisks) for different values their pairwise coupling parameters: in (a) setting all pairs in bistable regimes (when considered in isolation); or with one (b), two (c), or all three sets of coupling parameters (d) set in monostable regimes. In (a), the potential function landscape possesses four stable fixed points at in-phase and antiphase (white asterisks). In (b) with one dyadic relation (ϕ_1) in the monostable regime, four stable fixed points were also observed, two of them (grey asterisks) would not exist in their isolated dyad but are rescued by the coupling to the third oscillator. In (c) with two dyads in the monostable regime, three stable fixed points were found, including two (grey asterisks) that would not exist for their pairs in isolation. In (d), with all dyads in the monostable regime, all the antiphase regimes vanished, leaving a single stable fixed point with all oscillators coordinated in-phase.

with $(A_{\text{mon}}, B_{\text{mon}}) := (3, \frac{1}{2})$. The third triad has two of its dyads coupled monostably, while the third remains bistable:

$$(A_{01}, B_{01}) = (A_{02}, B_{02}) = (A_{\text{mon}}, B_{\text{mon}}) \quad \text{and} \quad (A_{12}, B_{12}) = (A_{\text{bi}}, B_{\text{bi}}). \quad (3.2c)$$

Finally, all three dyads are coupled monostably in the fourth dyad:

$$(A_{01}, B_{01}) = (A_{02}, B_{02}) = (A_{12}, B_{12}) = (A_{\text{mon}}, B_{\text{mon}}). \quad (3.2d)$$

Figure 2 shows heat map plots of the potentials in the four cases of eq. (3.2), with their local minima (within the region $-\frac{\pi}{2} \leq \phi_{1,2} \leq \frac{3\pi}{2}$) marked by asterisks. In particular, the gray asterisks in Figure 2(b) and (c) label stable fixed points in which dyads of oscillators that would be monostable in isolation nonetheless move anti-phase to one another. That is, in either of these cases, the coupling of an inherently monostable dyad to a third oscillator can render it functionally bistable. The only case in which no such stabilization occurs is that of Figure 2(d), wherein every dyad is coupled monostably.

The mechanism behind this phenomenon of *induced bistability* in dyads that would be monostable in isolation is not difficult to understand. If the coupling of either oscillator in such a dyad to a third oscillator is sufficiently deep in the bistable regime, then the attraction of one oscillator in the dyad to an antiphase motion with the third oscillator can overcome its repulsion to an antiphase motion with its dyadic partner. One can quantify concrete limits on the coupling parameters (A_{ij}, B_{ij}) needed to support this phenomenon by studying the eigenvalue problem for the linearized equations of motion derived from eq. (2.13).

4. Discussion

We have shown that dyads within a generalized HKB system [5] of three or more coupled oscillators can exhibit stable coordination patterns that their pairwise interaction would not otherwise allow. Specifically, dyads that would be only monostable in isolation can become bistable by virtue of their mutual coupling to a third oscillator. The attractors for the dynamics of a three-oscillator system can be identified easily using 2D visualizations of the interaction potential governing its dynamics. This work will help extend theoretical models of dynamic coordination in systems of multiple elements, for instance in sociophysics and neuroscience.

One potential application of this work is to gerontology. Sustaining antiphase motor coordination demands additional attentional resources [11], and becomes less intrinsically stable in older adults [12]. (See also [13] for experimental work involving younger adult subjects.) The combination of these empirical findings with the present theoretical work suggests possible ways to preserve and restore behavioral and social coordination in aging populations by employing third parties (social mates, caregivers, or therapists) to help individuals sustain a broader repertoire of coordinative abilities. This hypothesis, suggested by this theoretical model, still awaits empirical confirmation, which our team is currently pursuing in a clinical setting.

Another potential application is to neuroscience. The advent of functional connectivity methods [14] (aimed at quantifying transient, dynamical correlations in the activity of multiple brain regions) has emphasized that the coordination dynamics of neural ensembles cannot be fully explained by mere structural connectivity. Typical brain function at rest [15] routinely exhibits dynamics indicating shifting patterns of *functional* connectivity [16, 17]. However, the endogenous mechanisms by which some neural ensembles can influence the coordination dynamics of other, strongly and directly coupled ensembles remain obscure. Because of the empirical roots of the (generalized) HKB model in brain sciences [3], we anticipate that our work will help elucidate a mechanistic understanding of this phenomenon, and suggest precise quantitative tools to predict its occurrence.

Mathematically, Zhang's generalization of the HKB model to many oscillators [5] extends the widely studied Kuramoto model [18, 19], whose equations of motion coincide with eq. (2.3) with the "second-order" coupling switched off (i.e., with $B_{ij} = 0$). Eliminating these terms from the equations of motion, however, also eliminates the bistability that has been widely observed in behavioral, neural and social systems [2, 3, 13, 20]. Moreover, Zhang and colleagues [13, 5] have argued that the same "second-order" coupling in eq. (2.3) is indispensable for modeling certain empirical observations in the coordinative behaviors of larger groups of individuals. The present work offers a concrete realization of an additional principle, namely, that the larger context in which a (sub)system is embedded can profoundly affect the spectrum of coordination behaviors it can exhibit. In broader terms, this finding adds to a growing body of work illuminating a fundamental theme in complex systems theory, namely, that a system's whole generally differs from the sum of its parts.

Acknowledgments

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